1. Find the formulas for the differential area elements for:

   a) a plane (choose a easy one)

   b) a finite cylindrical shell (to the top/bottom and side separately),

   c) a spherical shell.

Solution The differential area element $d\vec{A}$, is found by taking the vector cross product of the differential displacements along the edges of a "box":
$d\vec{A} = d\vec{\ell}_1 \times d\vec{\ell}_2$

a) a plane

Use rectangular coordinates. In the xy-plane, the side lengths of the box are:

$$\begin{aligned} d\vec{\ell}_1 &= dx\,\hat{x} \\ d\vec{\ell}_2 &= dy\,\hat{y} \\ d\vec{A} &= d\vec{\ell}_1 \times d\vec{\ell}_2 \\ &= dx\,\hat{x} \times dy\,\hat{y} \\ &= dx\,dy\,\hat{z} \end{aligned}$$

If I did the cross product in the other order, the differential area vector would point in the opposite direction.

b) a finite cylindrical shell (including the area elements of the top and bottom),

Use cyclindrical coordinates.

I'll orient the cyclinder to the z-axis runs along the axis of symmetry of the cyclinder. The top and bottom are parallel to the $z = 0$ plane.

For the top of the cylindrical shell whose center:

$$\begin{aligned} d\vec{\ell}_1 &= ds\,\hat{s} \\ d\vec{\ell}_2 &= sd\phi\,\hat{\phi} \\ d\vec{A}_{top} &= d\vec{\ell}_1 \times d\vec{\ell}_2 \\ &= ds\,\hat{s} \times s\,d\phi\,\hat{\phi} \\ &= s\,ds\,d\phi\,\hat{z} \end{aligned}$$

I need to do the cross product in this order so that the differential area element points out of the closed cyclinder.

But from the bottom of the shell, I should do the cross product in the other order, which introduces a minus sign:

$$d\vec{A}_{bottom} = -s\,ds\,d\phi\,\hat{z}$$

For the side of the cyclindrical shell:

$$\begin{aligned} d\vec{\ell}_1 &= s\, d\phi\, \hat{\phi} \\ d\vec{\ell}_2 &= dz\, \hat{z} \\ d\vec{A}_{side} &= d\vec{\ell}_1 \times d\vec{\ell}_2 \\ &= s\, d\phi\, \hat{\phi} \times dz\, \hat{z} \\ &= s\, d\phi\, dz\, \hat{s} \end{aligned}$$

c) a spherical shell
Use spherical coordinates.

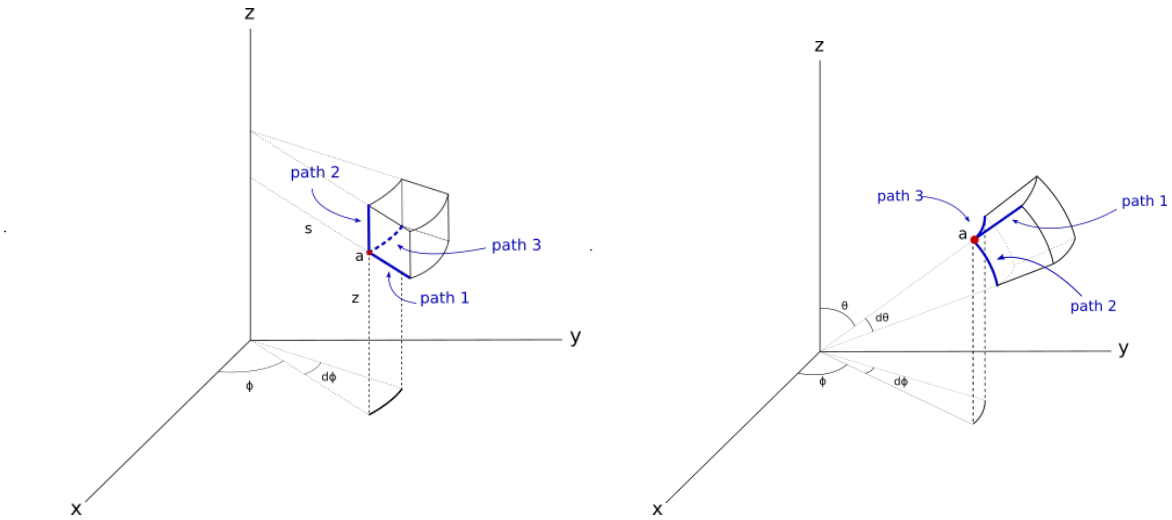
$$\begin{aligned} d\vec{\ell}_1 &= r\, d\theta\, \hat{\theta} \\ d\vec{\ell}_2 &= r \sin\theta\, d\phi\, \hat{\phi} \\ d\vec{A}_{side} &= d\vec{\ell}_1 \times d\vec{\ell}_2 \\ &= r\, d\theta\, \hat{\theta} \times r \sin\theta\, d\phi\, \hat{\phi} \\ &= r\ \sin\theta\, d\theta\, d\phi\, \hat{r} \end{aligned}$$

2. Find the formulas for the differential volume elements for:

   a) a thick slab

   b) a cylinder

   c) a sphere



Solution The differential volume element can be found by taking the triple product between differential displacements along the edges of a cube:

$$d\tau = (d\vec{\ell}_1 \times d\vec{\ell}_2) \cdot d\vec{\ell}_3$$

I have to evaluate the cross product first to avoid trying to take the cross product of a vector with a scalar.

- thick slab

  Use rectangular coordinates to describe a cube.

$$\begin{aligned} d\vec{\ell}_1 &= dx\,\hat{x} \\ d\vec{\ell}_2 &= dy\,\hat{y} \\ d\vec{\ell}_3 &= dz\,\hat{z} \\ d\tau_{slab} &= (d\vec{\ell}_1 \times d\vec{\ell}_2) \cdot d\vec{\ell}_3 \\ &= dx\,\hat{x} \times dy\,\hat{y} \cdot dz\,\hat{z} \\ &= dx\,dy\,\hat{z} \cdot dz\,\hat{z} \\ &= dx\,dy\,dz \end{aligned}$$

$$d\tau_{slab} = dx\,dy\,dz$$

- cylinder.

  Use cylindrical coordinates to describe a "pineapple chunk" cube.

$$\begin{aligned} d\vec{\ell}_1 &= ds\,\hat{s} \\ d\vec{\ell}_2 &= s\,d\phi\,\hat{\phi} \\ d\vec{\ell}_3 &= dz\,\hat{z} \\ d\tau_{cylinder} &= (d\vec{\ell}_1 \times d\vec{\ell}_2) \cdot d\vec{\ell}_3 \\ &= (ds\,\hat{s} \times s\,d\phi\,\hat{\phi}) \cdot dz\,\hat{z} \\ &= s\,ds\,d\phi\,\hat{z} \cdot dz\,\hat{z} \\ &= s\,ds\,d\phi\,dz \end{aligned}$$

$$d\tau_{cylinder} = s\,ds\,d\phi\,dz$$

- a sphere

  Use spherical coordinates to describe a "pumpkin piece" cube.

$$
\begin{aligned}
d\vec{\ell}_1 &= dr\,\hat{r} \\
d\vec{\ell}_2 &= r\,d\theta\,\hat{\theta} \\
d\vec{\ell}_3 &= r\,\sin\theta\,d\phi\,\hat{\phi} \\
d\tau_{sphere} &= (d\vec{\ell}_1 \times d\vec{\ell}_2) \cdot d\vec{\ell}_3 \\
&= (dr\,\hat{r} \times r\,d\theta\,\hat{\theta}) \cdot r\,\sin\theta\,d\phi\,\hat{\phi} \\
&= r\,dr\,d\theta\,\hat{\phi} \cdot r\,\sin\theta\,d\phi\,\hat{\phi} \\
&= r^2\,\sin\theta\,dr\,d\phi\,d\phi
\end{aligned}
$$

$$
d\tau_{sphere} = r^2\,\sin\theta\,dr\,d\phi\,d\phi
$$