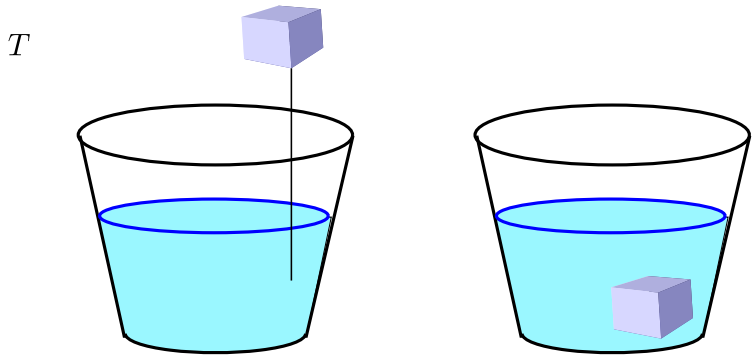



Figure 1: (a) We drop a small 350 K cube of metal into a bucket of water initially at 300 K. (b) After a short while, the metal has cooled to 346 K, and the water has warmed to 301 K.

Find $\Delta S_{\text{total}}$ as the system (the block and the water) goes from (a) to (b). Assume that it takes 2000 J of energy to raise the temperature of the water bath by 1 K.

**Solution** There are several perfectly good approaches, and I'm perfectly happy with any of the following approximate answers.

**Option 1** (decent approximation)

$$\Delta S_{\text{water}} + \Delta S_{\text{metal}} \approx \frac{2000 \text{ J}}{301K} - \frac{2000 \text{ J}}{346K} \tag{1}$$

$$= 0.86 \text{ J/K} \tag{2}$$

**Option 2** (decent approximation)

$$\Delta S_{\text{water}} + \Delta S_{\text{metal}} \approx \frac{2000 \text{ J}}{300K} - \frac{2000 \text{ J}}{350K} \tag{3}$$

$$= 0.95 \text{ J/K} \tag{4}$$

**Option 3** (best approximation yet)

$$\Delta S_{\text{water}} + \Delta S_{\text{metal}} \approx \frac{2000 \text{ J}}{300.5K} - \frac{2000 \text{ J}}{348K} \tag{5}$$

$$= 0.91 \text{ J/K} \tag{6}$$

Given that all the above are approximations, you might wonder how to find the exact answer (given the presumably approximate numbers such as 2000 J). We would do that by breaking the process into smaller subprocesses, so that each subprocess would have a smaller temperature change, and thus we could better justify using whichever of these three options we chose.

One way to do this would be to break the process into a finite number of smaller steps, e.g. you could heat the metal 500 J at a time:

$$\Delta S_{\text{metal}} \approx -\frac{500\text{ J}}{349.5\text{ K}} - \frac{500\text{ J}}{348.5\text{ K}} - \frac{500\text{ J}}{347.5\text{ K}} - \frac{500\text{ J}}{346.5\text{ K}} \tag{7}$$

$$= -5.747\text{ J/K} \tag{8}$$

In the same way we compute the change in entropy of the water

$$\Delta S_{\text{water}} \approx -\frac{500\text{ J}}{300.125\text{ K}} - \frac{500\text{ J}}{300.375\text{ K}} - \frac{500\text{ J}}{300.625\text{ K}} - \frac{500\text{ J}}{300.875\text{ K}} \tag{9}$$

$$= 6.655\text{ J/K} \tag{10}$$

$$\Delta S_{\text{water}} + \Delta S_{\text{metal}} \approx 0.91\text{J/K} \tag{11}$$

Note that I kept one extra digit on my intermediate answers, since I knew I'd be substracting two similar numbers, and wanted the rounding to work out all right. So this is the same as our "best approximation yet" above with this number of digits. So it may not have felt worth the effort.

But even so, you might not want to stop at four steps. We could instead break this process up into an infinite number of subprocesses by turning this sum into an integral. If you are a physics major, you'll be doing a lot of such integrals in your Energy and Entropy course.

To turn this into an integral, we have to figure out how the amount of heat transfered relates to a change in temperature. If we assume these are proportional (which is a reasonable approximation for moderate changes in temperature), we can obtain for the metal that the heat for a small step is $đQ = [500\text{ J/K}]dT$. Then we can integrate to find the change in entropy of the metal.

$$\Delta S_{\text{metal}} = \int_{350\text{ K}}^{346\text{ K}} \frac{dQ}{T} \tag{12}$$

$$= \int_{350\text{ K}}^{346\text{ K}} \frac{[500\text{ J/K}]dT}{T} \tag{13}$$

$$= [500\text{ J/K}]\ln T\Big|_{350\text{ K}}^{346\text{ K}} \tag{14}$$

$$= 500\ln\left(\frac{346}{350}\right)\text{ J/K} \tag{15}$$

$$= -5.747\text{ J/K} \tag{16}$$

In the water the entropy increases, and we can do the same kind of a calculation, but with $dQ = [2000\text{ J/K}]dT$, since the water changes temperature less.

$$\Delta S_{\text{water}} = \int_{300\text{ K}}^{301\text{ K}} \frac{[2000\text{ J/K}]dT}{T} \tag{17}$$

$$= 2000\ln\left(\frac{301}{300}\right)\text{ J/K} \tag{18}$$

$$= 6.656\text{ J/K} \tag{19}$$

$$\Delta S_{\text{water}} + \Delta S_{\text{metal}} \approx 0.91\text{ J/K} \tag{20}$$

So indeed our best approximation above was pretty good. If however we had been looking at a larger change in temperature (e.g. if they had reached equilibrium) the integral would have made a greater improvement.