

For the state

$$|\Psi\rangle = \sqrt{\frac{7}{10}}|2, 1, 0\rangle + \sqrt{\frac{1}{10}}|3, 2, 1\rangle + i\sqrt{\frac{2}{10}}|3, 1, 1\rangle$$

Calculate

- $\mathcal{P}(L_z = \hbar)$

Solution

$$\mathcal{P}_{L_z=\hbar} = |\langle 3, 2, 1|\Psi\rangle|^2 + |\langle 3, 1, 1|\Psi\rangle|^2 = \left|\sqrt{\frac{1}{10}}\right|^2 + \left|i\sqrt{\frac{2}{10}}\right|^2 = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \quad (1)$$

(2)

- $\langle L_z \rangle$

Solution There are a number of viable ways to find an expectation value, if the probabilities are easy to read off, I think summing the probabilities times the eigenvalues you care about is the easiest. We can read off the probabilities:

$$\mathcal{P}_{L_z=\hbar} = |\langle 3, 2, 1|\Psi\rangle|^2 + |\langle 3, 1, 1|\Psi\rangle|^2 = \left|\sqrt{\frac{1}{10}}\right|^2 + \left|i\sqrt{\frac{2}{10}}\right|^2 = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \quad (3)$$

$$\mathcal{P}_{L_z=0\hbar} = |\langle 2, 1, 0|\Psi\rangle|^2 = \left|\sqrt{\frac{7}{10}}\right|^2 = \frac{7}{10} \quad (4)$$

Now we can just take the weighted average:

$$\langle L_z \rangle = \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} P_{L_z=m\hbar} m\hbar = \left(\frac{7}{10}\right)(0\hbar) + \left(\frac{3}{10}\right)(\hbar) = \frac{3}{10}\hbar \quad (5)$$

Then, if you have time, continue with these calculations:

- $\mathcal{P}(L^2 = 2\hbar^2)$

Solution

$$\mathcal{P}_{L^2=2\hbar^2} = |\langle 2, 1, 0|\Psi\rangle|^2 + |\langle 3, 1, 1|\Psi\rangle|^2 = \left|\sqrt{\frac{7}{10}}\right|^2 + \left|i\sqrt{\frac{2}{10}}\right|^2 = \frac{7}{10} + \frac{2}{10} = \frac{9}{10} \quad (6)$$

(7)

- $\langle L^2 \rangle$

Solution Here we use the same strategy, but need to be aware of different degeneracies:

$$\mathcal{P}_{L^2=2\hbar^2} = |\langle 2, 1, 0 | \Psi \rangle|^2 + |\langle 3, 1, 1 | \Psi \rangle|^2 = \left| \sqrt{\frac{7}{10}} \right|^2 + \left| i\sqrt{\frac{2}{10}} \right|^2 = \frac{7}{10} + \frac{2}{10} = \frac{9}{10} \quad (8)$$

$$\mathcal{P}_{L^2=6\hbar^2} = |\langle 3, 2, 1 | \Psi \rangle|^2 = \left| \sqrt{\frac{1}{10}} \right|^2 = \frac{1}{10} \quad (9)$$

Now we can just take the weighted average:

$$\langle L^2 \rangle = \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} P_{L^2=\ell(\ell+1)\hbar^2} \ell(\ell+1)\hbar^2 = \left(\frac{9}{10} \right) (2\hbar^2) + \left(\frac{1}{10} \right) (6\hbar^2) = \frac{24}{10}\hbar^2 \quad (10)$$

- $\mathcal{P}(E = -13.6\text{eV}/3^2 = -1.51\text{eV})$

Solution

$$\mathcal{P}_{E=-13.6\text{eV}/9} = |\langle 3, 2, 1 | \Psi \rangle|^2 + |\langle 3, 1, 1 | \Psi \rangle|^2 = \left| \sqrt{\frac{1}{10}} \right|^2 + \left| i\sqrt{\frac{2}{10}} \right|^2 = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \quad (11)$$

$$(12)$$

- $\langle E \rangle$

Solution

$$\mathcal{P}_{E=-13.6\text{eV}/9} = |\langle 3, 2, 1 | \Psi \rangle|^2 + |\langle 3, 1, 1 | \Psi \rangle|^2 = \left| \sqrt{\frac{1}{10}} \right|^2 + \left| i\sqrt{\frac{2}{10}} \right|^2 = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \quad (13)$$

$$\mathcal{P}_{E=-13.6\text{eV}/4} = |\langle 2, 1, 0 | \Psi \rangle|^2 = \left| \sqrt{\frac{7}{10}} \right|^2 = \frac{7}{10} \quad (14)$$

Now we can just take the weighted average:

$$\langle E \rangle = \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} P_{E=-13.6\text{eV}/n^2} \left(-\frac{13.6\text{eV}}{n^2} \right) = \left(\frac{7}{10} \right) \left(-\frac{13.6\text{eV}}{4} \right) + \left(\frac{3}{10} \right) \left(-\frac{13.6\text{eV}}{9} \right) = -2.83\text{eV} \quad (15)$$

- What measurements can be degenerate on the Hydrogen atom?

Solution Measurements of the z-component of angular momentum, total angular momentum, and energy can all be degenerate on the hydrogen atom. When we have 3 quantum numbers we can always lock one down and freely change the other two to create different eigenstates which will be degenerate for the locked quantum number.

- What is the time development of this state?

Solution Since the hydrogen atom energy eigenvalues depend only on n ,

$$E_n = -\frac{13.6\text{eV}}{n^2}, \quad (16)$$

each energy eigenstate picks up a time-dependent phase factor $e^{-iE_n t/\hbar}$. Therefore,

$$|\Psi(t)\rangle = \sqrt{\frac{7}{10}}e^{-iE_2 t/\hbar}|2, 1, 0\rangle + \sqrt{\frac{1}{10}}e^{-iE_3 t/\hbar}|3, 2, 1\rangle + i\sqrt{\frac{2}{10}}e^{-iE_3 t/\hbar}|3, 1, 1\rangle. \quad (17)$$

Using

$$E_2 = -\frac{13.6\text{eV}}{4}, \quad E_3 = -\frac{13.6\text{eV}}{9}, \quad (18)$$

we can also write

$$|\Psi(t)\rangle = \sqrt{\frac{7}{10}}e^{-iE_2 t/\hbar}|2, 1, 0\rangle + e^{-iE_3 t/\hbar} \left(\sqrt{\frac{1}{10}}|3, 2, 1\rangle + i\sqrt{\frac{2}{10}}|3, 1, 1\rangle \right). \quad (19)$$

Notice that the two $n = 3$ states have the same energy, so they acquire the same overall time-dependent phase.

- What is the probability of finding the particle in the region $0 < \theta < \pi/6$, $\pi/3 < \phi < \pi/2$, and $r_1 < r < r_2$?

Solution To find the probability of locating the particle in a particular region of space, we need to integrate the probability density over that region:

$$\mathcal{P} = \int_{r_1}^{r_2} \int_0^{\pi/6} \int_{\pi/3}^{\pi/2} |\Psi(r, \theta, \phi, t)|^2 r^2 \sin \theta d\phi d\theta dr. \quad (20)$$

For this state, the position-space wavefunction is

$$\Psi(r, \theta, \phi, t) = \sqrt{\frac{7}{10}}e^{-iE_2 t/\hbar}R_{21}(r)Y_1^0(\theta, \phi) \quad (21)$$

$$+ \sqrt{\frac{1}{10}}e^{-iE_3 t/\hbar}R_{32}(r)Y_2^1(\theta, \phi) \quad (22)$$

$$+ i\sqrt{\frac{2}{10}}e^{-iE_3 t/\hbar}R_{31}(r)Y_1^1(\theta, \phi). \quad (23)$$