Consider 5 (distinguishable beads) arranged in a ring.

1. Construct a matrix operator that translates (moves) each beads forward one "bead slot" along the ring (counterclockwise).

Solution I want to move each bead once counterclockwise, or one up the column vector, so I need to have a matrix with only non-zero entries on the spaces just above the main diagonal (plus one in the off corner since I have a periodic boundary condition). So I have:

$$S^{\uparrow} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can test this on a arbitrary column vector:

$$S^{\uparrow} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} b \\ c \\ d \\ e \\ a \end{bmatrix}$$

Yep it moves each element one space up!

2. Write down an eigenvalue equation for this operator.

3. How can you find eigenvalues and eigenvectors of this operator? Can you come up with two different approaches??

Solution If I operate this matrix on a column vector 5 times, I get the exact same vector back again, so an eigenvalue equation is:

$$(S^{\uparrow})^5 \vec{s} = 1\vec{s} \tag{1}$$

But this isn't an eigenvalue equation for $S^{\uparrow}$, it is an eigevalue equation for $(S^{\uparrow})^5$. However, I do know if I use $S^{\uparrow}$ on a column vector 5 times, I will get the eigenvalue of 1, and since I know I need the same eigenvalue for $S^{\uparrow}$ each time, I can take the 5th root of 1 and use it as the eigenvalue of $S^{\uparrow}$. Thus, I take:

$$\sqrt[5]{1} = (1)^{1/5} = \left(e^{i2\pi n}\right)^{1/5} = e^{i\frac{2\pi}{5}n}$$

Where I used the exponential complex representation of 1 with a integer n which goes from 1 to 5 (because I have 5 beads). So my eigenvalues are, in exponential and retanglar complex numbers:

n	1	2	3	4	5
Exponential	$e^{i\frac{2\pi}{5}}$	$e^{i\frac{4\pi}{5}}$	$e^{i\frac{6\pi}{5}}$	$e^{i\frac{8\pi}{5}}$	$\cancelto{1}{e^{i\frac{10\pi}{5}}}$

Solution Now I want the eigenvectors. To do this, I'll use some conceptual reasoning about what the transformation does.
I'll start with thinking about what $S^{\uparrow}$ does to a vector:

$$S^{\uparrow}\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} b \\ c \\ d \\ e \\ a \end{bmatrix}$$

Solution For eigenvalue $\lambda_5 = 1$, the eigenvalue equation is:

$$S^{\uparrow}\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = 1\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

This means that:

$$\begin{aligned} b &= a \\ c &= b \\ d &= c \\ e &= d \\ a &= e \end{aligned}$$

This only works if $a = b = c = d = e$, and the normalized eigenvector is:

$$|1\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution For eigenvalue $\lambda_1 = e^{i\frac{2\pi}{5}}$, the eigenvalue equation is:

$$S^{\uparrow}\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = e^{i\frac{2\pi}{5}}\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

This means that:

$$\begin{bmatrix} b \\ c \\ d \\ e \\ a \end{bmatrix} = e^{i\frac{2\pi}{5}}\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

$$b = e^{i\frac{2\pi}{5}}a$$
$$c = e^{i\frac{2\pi}{5}}b$$
$$d = e^{i\frac{2\pi}{5}}c$$
$$e = e^{i\frac{2\pi}{5}}d$$
$$a = e^{i\frac{2\pi}{5}}e$$

Solution This works if:

$$\left|e^{i\frac{2\pi}{5}}\right\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ e^{i\frac{2\pi}{5}} \\ e^{i\frac{4\pi}{5}} \\ e^{i\frac{6\pi}{5}} \\ e^{i\frac{8\pi}{5}} \end{bmatrix}$$

Using the same reasoning for the rest of the eigenvalues:

$$\left|e^{i\frac{4\pi}{5}}\right\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ e^{i\frac{4\pi}{5}} \\ e^{i\frac{8\pi}{5}} \\ e^{i\frac{12\pi}{5}} \\ e^{i\frac{16\pi}{5}} \end{bmatrix} \quad \left|e^{i\frac{6\pi}{5}}\right\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ e^{i\frac{6\pi}{5}} \\ e^{i\frac{12\pi}{5}} \\ e^{i\frac{18\pi}{5}} \\ e^{i\frac{24\pi}{5}} \end{bmatrix} \quad \left|e^{i\frac{8\pi}{5}}\right\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ e^{i\frac{8\pi}{5}} \\ e^{i\frac{16\pi}{5}} \\ e^{i\frac{24\pi}{5}} \\ e^{i\frac{32\pi}{5}} \end{bmatrix}$$

Go ahead and check that these vectors solve the eigenvalue equations for their corresponding eigenvalue.