

Calculate the momentum space wave function for a particle in an energy eigenstate of the infinite square well.

Solution

$$\begin{aligned}\psi_E(p) &= \langle p | E_n \rangle = \langle p | \left(\int_{-\infty}^{\infty} |x\rangle \langle x| dx \right) | E_n \rangle \\ &= \int_{-\infty}^{\infty} \langle p | x \rangle \langle x | E_n \rangle dx \\ &= \int_{-\infty}^{\infty} \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \psi_E(x) dx\end{aligned}$$

The energy eigenstates are:

$$\psi_E(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} & 0 < x < L \\ 0 & \text{else} \end{cases}$$

So, my transformation becomes:

$$\begin{aligned}\langle p | \psi \rangle = \phi(p) &= \int_{-\infty}^0 0 dx + \int_0^L \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \sin \frac{n\pi x}{L} dx + \int_0^{\infty} 0 dx \\ &= \int_0^L \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \sin \frac{n\pi x}{L} dx \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_0^L e^{-ipx/\hbar} \left(\frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} \right) dx\end{aligned}$$

Let $\frac{n\pi}{L} = \frac{p_0}{\hbar}$:

$$\begin{aligned}
 \langle p|\psi\rangle &= \phi(p) = \frac{1}{2i\sqrt{2\pi\hbar}} \int_0^L e^{-ipx/\hbar} \left(e^{ip_0x/\hbar} - e^{-ip_0x/\hbar} \right) dx \\
 &= \frac{1}{2i\sqrt{2\pi\hbar}} \int_0^L \left(e^{-i(p-p_0)x/\hbar} - e^{-i(p+p_0)x/\hbar} \right) dx \\
 &= \frac{1}{2\sqrt{2\pi\hbar}} \left(\frac{e^{-i(p-p_0)L/\hbar}}{-i(p-p_0)/\hbar} \Big|_0^L - \frac{e^{-i(p+p_0)L/\hbar}}{-i(p+p_0)/\hbar} \Big|_0^L \right) \\
 &= \sqrt{\frac{\hbar}{8\pi}} \left(\frac{e^{-i(p-p_0)L/\hbar} - 1}{(p-p_0)} - \frac{e^{-i(p+p_0)L/\hbar} - 1}{(p+p_0)} \right)
 \end{aligned}$$

This is not a simple sum of momentum eigenstates.

To put this into a slightly better form, I'll use the fact that:

$$\begin{aligned}
 e^{-i\theta} - 1 &= e^{-i\theta/2} (e^{-i\theta/2} - e^{+i\theta/2}) \\
 &= e^{-i\theta/2} (-2i \sin \frac{\theta}{2}) \\
 &= -i2e^{-i\theta/2} \sin \frac{\theta}{2}
 \end{aligned}$$

Making the substitution:

$$\phi(p) = \frac{-2i\hbar}{\sqrt{8\pi\hbar}} \left(\frac{e^{-i(p-p_0)L/2\hbar} \sin \frac{(p-p_0)L}{2\hbar}}{p-p_0} - \frac{e^{-i(p+p_0)L/2\hbar} \sin \frac{(p+p_0)L}{2\hbar}}{p+p_0} \right)$$

To better understand this, I'll calculate the probability density and plot it:

$$\begin{aligned}
 |\phi(p)|^2 &= \frac{\hbar}{2\pi} \left| \frac{e^{-i(p-p_0)L/2\hbar} \sin \frac{(p-p_0)L}{2\hbar}}{p-p_0} - \frac{e^{-i(p+p_0)L/2\hbar} \sin \frac{(p+p_0)L}{2\hbar}}{p+p_0} \right|^2 \\
 &= \frac{\hbar}{2\pi} \left(\frac{\sin^2 \frac{(p-p_0)L}{2\hbar}}{(p-p_0)^2} + \frac{\sin^2 \frac{(p+p_0)L}{2\hbar}}{(p+p_0)^2} \right. \\
 &\quad - \frac{e^{-i(p-p_0)L/2\hbar} e^{+i(p+p_0)L/2\hbar} \sin \frac{(p-p_0)L}{2\hbar} \sin \frac{(p+p_0)L}{2\hbar}}{(p-p_0)(p+p_0)} \\
 &\quad \left. - \frac{e^{+i(p-p_0)L/2\hbar} e^{-i(p+p_0)L/2\hbar} \sin \frac{(p-p_0)L}{2\hbar} \sin \frac{(p+p_0)L}{2\hbar}}{(p-p_0)(p+p_0)} \right) \\
 &= \frac{\hbar}{2\pi} \left[\frac{\sin^2 \frac{(p-p_0)L}{2\hbar}}{(p-p_0)^2} + \frac{\sin^2 \frac{(p+p_0)L}{2\hbar}}{(p+p_0)^2} \right. \\
 &\quad \left. - \frac{\cos \frac{p_0 L}{\hbar} \sin \frac{(p-p_0)L}{2\hbar} \sin \frac{(p+p_0)L}{2\hbar}}{(p-p_0)(p+p_0)} \right]
 \end{aligned}$$

Remembering that $p_0 = \frac{n\pi\hbar}{L}$, the cosine term simplifies:

$$\begin{aligned}
 |\phi(p)|^2 &= \frac{\hbar}{2\pi} \left[\frac{\sin^2 \frac{(p-p_0)L}{2\hbar}}{(p-p_0)^2} + \frac{\sin^2 \frac{(p+p_0)L}{2\hbar}}{(p+p_0)^2} \right. \\
 &\quad \left. - \cos n\pi \frac{\sin \frac{(p-p_0)L}{2\hbar} \sin \frac{(p+p_0)L}{2\hbar}}{(p-p_0)(p+p_0)} \right]
 \end{aligned}$$

Plotting the probability density, (setting $\hbar = 1$, $L = 1$, and $n = 1$) I see two strong peaks at the $\pm p_0$ but not quite delta functions.

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