The Electrostatic Field Due to a Ring of Charge

1. Use Coulomb's law
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')\,(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau'$$
to find the electric field everywhere in space due to a charged ring with radius R and total charge Q.

Solution Going through the process again with the electric field, we take inventory about what is the same and what is different from our calculation for the electric potential. Coulomb's law is:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')\,(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau' \tag{1}$$

The geometry of our ring is the same, as is the charge, so we can immediately say:

$$\rho' d\tau' \rightarrow \lambda' d\ell' \rightarrow = \frac{Q}{2\pi} d\phi' \tag{2}$$

$$|\vec{r}-\vec{r}'| = \sqrt{s^2+R^2-2sR\cos(\phi-\phi')+z^2} \tag{3}$$

Of course we will have a different power of the magnitude in the denominator, but besides that, these elements are the same as for the potential, so we have:

$$\vec{E}(\vec{r}) = \frac{Q}{8\pi^2\epsilon_0}\int \frac{(\vec{r}-\vec{r}')\,d\phi'}{(s^2+R^2-2sR\cos(\phi-\phi')+z^2)^{\frac{3}{2}}} \tag{4}$$

So we just have this $(\vec{r}-\vec{r}')$ element to deal with. Of particular importance, this is where our vector directional components come from, which we need to describe the electric field. We have a few choices we could make here, but we need to think about integrating these vectors, and if we were to write $\vec{r}'$ in cylindrical coordinates, we run into an issue which is that $\hat{s}'$ and $\hat{\phi}'$ change as we move around in space (imagine laying on the positive x-axis where $\hat{s}$ points in $\hat{x}$, whereas if we move to the positive y-axis, $\hat{s}$ points in $\hat{y}$). Therefore, we want to write these in Cartesian basis vectors (these basis vectors point in the same direction regardless of where you are located in space, so $\hat{x}$ will always point in the same direction everywhere on the ring), and we will translate the magnitudes into cylindrical coordinates so we can have them match the rest of our equation!

So we will write out our distance vectors. Remember $\vec{r}$ points to the place in space we care about measuring the electric field, which for the time being is anywhere, so we can leave it generalized. $\vec{r}'$ points towards the charge, so we need it to point to every spot on the ring, which is a distance R from the origin and lays only in the xy-plane, so $z'=0$. And we have:

$$\vec{r} = x\hat{x}+y\hat{y}+z\hat{z} \tag{5}$$
$$\vec{r}' = x'\hat{x}+y'\hat{y}+(0)\;\hat{z} \tag{6}$$

NOTE: Here I did not put primes on the Cartesian basis vectors, and the reason is because they don't change as we move in space, $\hat{x}' = \hat{x}$ everywhere and always, and the same for y and z. This is NOT the same for $\hat{s}, \hat{\phi}, \hat{r}$, or $\hat{\theta}$ where the prime and un-primed distinction will be critical!

From here we use our coordinate transformations to change these vectors into cylindrical coordinates so we can integrate in cylindrical coordinates:

$$x = s\cos(\phi) \tag{7}$$
$$y = s\sin(\phi) \tag{8}$$
$$z = z \tag{9}$$

The primed magnitudes will be the same, except every coordinate will be primed, so we have:

$$\vec{r} = s\cos(\phi)\hat{x} + s\sin(\phi)\hat{y} + z\hat{z} \tag{10}$$
$$\vec{r}' = s'\cos(\phi')\hat{x} + s'\sin(\phi')\hat{y} \tag{11}$$

Now we recognize that $s' = R$ since the ring's geometry is always a fixed horizontal distance R from the origin, and we can now subtract these to get the piece we need for our overall equation:

$$\vec{r} - \vec{r}' = (s\cos(\phi) - R\cos(\phi'))\,\hat{x} + (s\sin(\phi) - R\sin(\phi'))\,\hat{y} + z\hat{z} \tag{12}$$

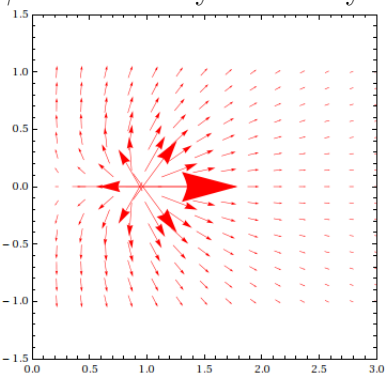
Now we put this piece back into our overall equation:

$$\vec{E}(\vec{r}) = \frac{Q}{8\pi^2\epsilon_0}\int \frac{((s\cos(\phi) - R\cos(\phi'))\,\hat{x} + (s\sin(\phi) - R\sin(\phi'))\,\hat{y} + z\hat{z})}{(s^2 + R^2 - 2sRcos(\phi - \phi') + z^2)^{\frac{3}{2}}}d\phi' \tag{13}$$

We can separate this into 3 different integrals based on the basis vectors:

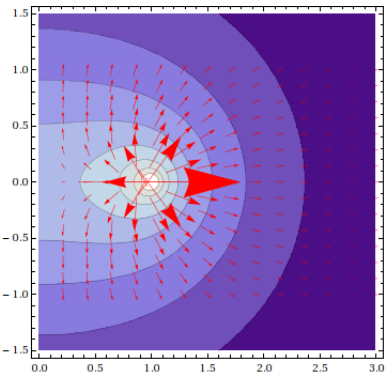
$$\vec{E}(\vec{r}) = \frac{Q}{8\pi^2\epsilon_0}\int \frac{(s\cos(\phi) - R\cos(\phi'))\,\hat{x}}{(s^2 + R^2 - 2sR\cos(\phi - \phi') + z^2)^{\frac{3}{2}}}d\phi' \tag{14}$$
$$+\frac{Q}{8\pi^2\epsilon_0}\int \frac{(s\sin(\phi) - R\sin(\phi'))\,\hat{y}}{(s^2 + R^2 - 2sR\cos(\phi - \phi') + z^2)^{\frac{3}{2}}}d\phi' \tag{15}$$
$$+\frac{Q}{8\pi^2\epsilon_0}\int \frac{z\hat{z}}{(s^2 + R^2 - 2sR\cos(\phi - \phi') + z^2)^{\frac{3}{2}}}d\phi' \tag{16}$$

From here, we can do various types of approximations to do these integrals by hand.
To visualize this solution, it is sufficient to plot the value of the potential in a plane of constant ϕ due to the cylindrical symmetry of the answer.



If we plot the electric field on top of the electrostatic potential, we can see that the first is the (negative) gradient of the second:

$$\vec{E} = -\vec{\nabla} V$$



2. Evaluate your expression for the special case that $\vec{r}$ is on the z-axis.

Solution Field points on the z-axis have $s = 0$, which simplifies the expression to:

$$\vec{E}(\vec{r}) = \frac{Q}{8\pi^2\epsilon_0}\int \frac{(-R\cos(\phi'))\,\hat{x}}{(R^2+z^2)^{\frac{3}{2}}}d\phi' \tag{17}$$

$$+\frac{Q}{8\pi^2\epsilon_0}\int \frac{(-R\sin(\phi'))\,\hat{y}}{(R^2+z^2)^{\frac{3}{2}}}d\phi' \tag{18}$$

$$+\frac{Q}{8\pi^2\epsilon_0}\int \frac{z\hat{z}}{(R^2+z^2)^{\frac{3}{2}}}d\phi' \tag{19}$$

The denominator is a constant of the integration and pulls out:

$$\vec{E}(\vec{r}) = \frac{Q}{8\pi^2\epsilon_0}\frac{1}{(R^2+z^2)^{\frac{3}{2}}}\left[\int (-R\cos(\phi'))\,\hat{x}d\phi' + \int (-R\sin(\phi'))\,\hat{y}d\phi' + \int z\hat{z}d\phi'\right] \tag{20}$$

3. Evaluate your expression for the special case that $\vec{r}$ is on the x-axis.