Outer Product of a Vector on Itself

1. Your group will be given a pair (or triple) of vectors below, find the matrix that is the outer product of each vector on itself (i.e., $|v_1\rangle\langle v_1|$)? All the vectors are written in the S_z basis.

1) $|+\rangle \doteq \begin{bmatrix}1\\0\end{bmatrix}$ $\quad |-\rangle \doteq \begin{bmatrix}0\\1\end{bmatrix}$

2) $|+\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ $\quad |-\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

3) $|+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$ $\quad |-\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$

4) $|v_7\rangle \doteq \frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}$ $\quad |v_8\rangle \doteq \frac{1}{5}\begin{bmatrix}4\\-3\end{bmatrix}$

5) $|v_9\rangle \doteq \begin{bmatrix}a\\be^{i\phi}\end{bmatrix}$ $\quad |v_{10}\rangle \doteq \begin{bmatrix}b\\-ae^{i\phi}\end{bmatrix}$

6) $|1\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\1\\\frac{1}{\sqrt{2}}\end{bmatrix}$ $\quad |0\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\quad |-1\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\-1\\\frac{1}{\sqrt{2}}\end{bmatrix}$

2. What is the square of each of your outer products?

Solution All of the matrices square to themselves. This makes sense for a projection operator. If I interpret squaring as doing the projection twice, then after the first projection happens, the second projection can't change the vector - it's already projected into the correct direction.

3. What is the product of each pair of your outer products?

Solution All of these products are zero.

4. For each row of vectors, add all of the outer products.

Solution In each case you get the identity matrix.

Solution These questions are easier in Dirac notation!
Take a generic normalized vector $|n\rangle$. Make the projection operator $|n\rangle\langle n|$ and let it act on another generic vector $|\psi\rangle$:

$$|n\rangle\langle n|\psi\rangle = \langle n|\psi\rangle\,|n\rangle$$

Which is a vector in the $|n\rangle$ direction.
To square:

$$\begin{aligned}(|n\rangle\langle n|)(|n\rangle\langle n|) &= |n\rangle\left(\cancelto{1}{\langle n|n\rangle}\right)\langle n| \\ &= |n\rangle\langle n|\end{aligned}$$

5. What is the determinant of each of your outer products?

Solution All of the determinants are 0.

6. What is the transformation caused by each of your outer products?

   Bonus: How would you answer questions (2), (3), (4) staying purely in Dirac bra-ket notation?