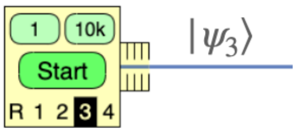
Finding the Unknown States Leaving an Oven (Spin-$\frac{1}{2}$)

1. Launch the Spins Laboratory simulation and choose Unknown # 3 on the oven. This causes the atoms to come out of the oven in a definite quantum state (instead of a random state), which we call $|\psi_3\rangle$.



2. Assume that we want to write the unknown state vectors in terms of the $|\pm\rangle$ basis, *i.e.* $|\psi_3\rangle = a|+\rangle + be^{i\gamma}|-\rangle$, where a and b are real. We thus must use the data to find the values of a, b and γ. Measure the six probabilities $|_n\langle\pm|\psi_3\rangle|^2$, where $|\pm\rangle_n$ corresponds to the spin states ("spin up" and "spin down") along the three axes $n = x, y$ or z. Fill in the table for $|\psi_3\rangle$ on the worksheet.

3. Using the probabilities determined from your experiments, calculate Unknown #3 ($|\psi_3\rangle$) and write it in the $|\pm\rangle$ basis.

4. Using the SPINS program, design and run a simulated experiment to verify your calculated state (Hint: recall the general spin-$\frac{1}{2}$ state vector can be written as $|+\rangle_n = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$).

5. Repeat this exercise for Unknown # 4 ($|\psi_4\rangle$).

Unknown $|\psi_3\rangle$

Probabilities	Spin Component		
Result	S_x	S_y	S_z
$\mathcal{P}(\frac{\hbar}{2})$			
$\mathcal{P}(-\frac{\hbar}{2})$			

Solution Probabilities:
1/2, 97/100, 1/3
1/2, 3/100, 2/3,

$$|\psi_3\rangle = \sqrt{\frac{1}{3}}\,|+\rangle + \sqrt{\frac{2}{3}}e^{i\pi/2}\,|-\rangle$$

Explanation
I can write the state generically as:

$$|\psi_4\rangle = a\,|+\rangle_z + be^{i\phi}\,|-\rangle_z$$

where a and b are real.
The probabilities of the S_z component of spin tell me a and b:

$$\begin{aligned}
\mathcal{P}(S_z = \hbar/2) &= \frac{1}{3} \\
|_z\langle +|\psi_3\rangle|^2 &= |a|^2 \\
\rightarrow a &= \frac{1}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(S_z = -\hbar/2) &= \frac{2}{3} \\
|_z\langle -|\psi_3\rangle|^2 &= |be^{i\phi}|^2 = |b|^2 \\
\rightarrow b &= \sqrt{\frac{2}{3}}
\end{aligned}$$

To find the relative phase, I'll look at the probabilities for the S_x and S_y components:

$$\begin{aligned}
\mathcal{P}(S_x = \hbar/2) &= \frac{1}{2} \\
|_x\langle +|\psi_3\rangle|^2 &= \left|\left(\frac{1}{\sqrt{2}}\,{}_z\langle +| + \frac{1}{\sqrt{2}}\,{}_z\langle -|\right)\left(\frac{1}{\sqrt{3}}\,|+\rangle_z + \sqrt{\frac{2}{3}}e^{i\phi}\,|-\rangle_z\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{6}} + \sqrt{\frac{2}{6}}e^{i\phi}\right|^2 \\
&= \frac{1}{6}\left(1 + \sqrt{2}e^{i\phi}\right)\left(1 + \sqrt{2}e^{-i\phi}\right) \\
&= \frac{1}{6}\left(1 + \sqrt{2}(e^{i\phi} + e^{-i\phi}) + 2\right) \\
&= \frac{1}{6}\left(3 + 2\sqrt{2}\cos\phi\right) \\
\frac{1}{2} &= \frac{1}{2} + \frac{\sqrt{2}}{3}\cos\phi \\
0 &= \cos\phi \\
\phi &= \pm\frac{\pi}{2}
\end{aligned}$$

This gives me two possible phase angles. Now try the an S_y probability:

$$\begin{aligned}
\mathcal{P}(S_y = \hbar/2) &= \frac{97}{100} \\
|_y\langle +|\psi_3\rangle|^2 &= \left| \left(\frac{1}{\sqrt{2}}\,{}_z\langle +| - \frac{i}{\sqrt{2}}\,{}_z\langle -| \right) \left(\frac{1}{\sqrt{3}}|+\rangle_z + \sqrt{\frac{2}{3}}e^{i\phi}|+\rangle_z \right) \right|^2 \\
&= \left| \frac{1}{\sqrt{6}} + i\sqrt{\frac{2}{6}}e^{i\phi} \right|^2 \\
&= \frac{1}{6}\left(1 + i\sqrt{2}e^{i\phi}\right)\left(1 - i\sqrt{2}e^{-i\phi}\right) \\
&= \frac{1}{6}\left(1 + i\sqrt{2}(e^{i\phi} - e^{-i\phi}) + 2\right) \\
&= \frac{1}{6}\left(3 + 2\sqrt{2}\sin\phi\right) \\
\frac{97}{100} &= \frac{1}{2} + \frac{\sqrt{2}}{3}\sin\phi \\
\frac{47}{100} &= \frac{\sqrt{2}}{3}\sin\phi \\
\sin\phi &= \frac{(3)(47)}{100\sqrt{2}} \\
\sin\phi &\approx 1 \\
\phi &= +\frac{\pi}{2}
\end{aligned}$$

Unknown $|\psi_4\rangle$

Probabilities	Spin Component		
Result	S_x	S_y	S_z
$\mathcal{P}(\frac{\hbar}{2})$			
$\mathcal{P}(-\frac{\hbar}{2})$			

Solution Probabilities:
1/2, 14/15, 1/4
1/2, 1/15, 3/4,

$$|\psi_4\rangle = \frac{1}{2}|+\rangle + \frac{\sqrt{3}}{2}e^{i\pi/2}|-\rangle$$