1 Spin Matrix

The Pauli spin matrices σ_x, σ_y, and σ_z are defined by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices are related to angular momentum in quantum mechanics.

(a) By drawing pictures, convince yourself that the arbitrary unit vector $\hat{n}$ can be written as:

$$\hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

where θ and ϕ are the parameters used to describe spherical coordinates.

(b) Find the entries of the matrix $\hat{n}\cdot\vec{\sigma}$ where the “matrix-valued-vector” $\vec{\sigma}$ is given in terms of the Pauli spin matrices by

$$\vec{\sigma} = \sigma_x\,\hat{x} + \sigma_y\,\hat{y} + \sigma_z\,\hat{z}$$

and $\hat{n}$ is given in part (a) above.

2 Eigenvectors of Pauli Matrices

Do this easy problem OR Eigen Spin Challenge - but not both.

(a) Find the eigenvalues and normalized eigenvectors of the Pauli matrices σ_x, σ_y, and σ_z (see the Spins Reference Sheet posted on the course website).

3 Eigen Spin Challenge

Optional Challenging Alternative to Eigenvectors of Pauli Matrices.

Consider the arbitrary Pauli matrix $\sigma_n = \hat{n}\cdot\vec{\sigma}$ where $\hat{n}$ is the unit vector pointing in an arbitrary direction.

(a) Find the eigenvalues and normalized eigenvectors for σ_n. The answer is:

$$\begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{pmatrix} \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{pmatrix}$$

It is not sufficient to show that this answer is correct by plugging into the eigenvalue equation. Rather, you should do all the steps of finding the eigenvalues and eigenvectors as if you don't know the answer. Hint: $\sin\theta = \sqrt{1-\cos^2\theta}$.

(b) Show that the eigenvectors from part (a) above are orthogonal.

(c) Simplify your results from part (a) above by considering the three separate special cases: $\hat{n} = \hat{\imath}$, $\hat{n} = \hat{\jmath}$, $\hat{n} = \hat{k}$. In this way, find the eigenvectors and eigenvalues of σ_x, σ_y, and σ_z.

4 Completeness Relation Change of Basis

(a) Given the polar basis kets written as a superposition of Cartesian kets

$$\begin{aligned} |\hat{s}\rangle &= \cos\phi\,|\hat{x}\rangle + \sin\phi\,|\hat{y}\rangle \\ \left|\hat{\phi}\right\rangle &= -\sin\phi\,|\hat{x}\rangle + \cos\phi\,|\hat{y}\rangle \end{aligned}$$

Find the following quantities:

$$\langle\hat{x}|\hat{s}\rangle\,, \quad \langle\hat{y}|\hat{s}\rangle\,, \quad \left\langle\hat{x}\middle|\hat{\phi}\right\rangle, \quad \left\langle\hat{y}\middle|\hat{\phi}\right\rangle$$

(b) Given a vector written in the polar basis

$$|\vec{v}\rangle = a\,|\hat{s}\rangle + b\left|\hat{\phi}\right\rangle$$

where a and b are known. Find coefficients c and d such that

$$|\vec{v}\rangle = c\,|\hat{x}\rangle + d\,|\hat{y}\rangle$$

Do this by using the completeness relation:

$$|\hat{x}\rangle\langle\hat{x}| + |\hat{y}\rangle\langle\hat{y}| = 1$$

(c) Using a completeness relation, change the basis of the spin-1/2 state

$$|\Psi\rangle = g\,|+\rangle + h\,|-\rangle$$

into the S_y basis. In otherwords, find j and k such that

$$|\Psi\rangle = j\,|+\rangle_y + k\,|-\rangle_y$$

5 Spin One Eigenvectors

The operator $\hat{S}_x$ for spin-1 (in the z-basis) may be written as:

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) Find the eigenvalues of this matrix.

(b) Find the eigenvectors corresponding to each eigenvalue.

6 Diagonalization

(a) Let

$$|\alpha\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \text{and} \qquad |\beta\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

Show that $|\alpha\rangle$ and $|\beta\rangle$ are orthonormal. (If a pair of vectors is orthonormal, that suggests that they might make a good basis.)

(b) Consider the matrix

$$C \doteq \begin{pmatrix}3 & 1\\1 & 3\end{pmatrix}$$

Show that the vectors $|\alpha\rangle$ and $|\beta\rangle$ are eigenvectors of C and find the eigenvalues. (Note that showing something is an eigenvector of an operator is far easier than finding the eigenvectors if you don't know them!)

(c) A operator is always represented by a diagonal matrix if it is written in terms of the basis of its own eigenvectors. What does this mean? Find the matrix elements for a new matrix E that corresponds to C expanded in the basis of its eigenvectors, i.e. calculate $\langle\alpha|C|\alpha\rangle$, $\langle\alpha|C|\beta\rangle$, $\langle\beta|C|\alpha\rangle$ and $\langle\beta|C|\beta\rangle$ and arrange them into a sensible matrix E. Explain why you arranged the matrix elements in the order that you did.

(d) Find the determinants of C and E. How do these determinants compare to the eigenvalues of these matrices?

7 Diagonalization Part II

(Optional, ungraded: Only do this problem if you are comfortable with mathematician's linear algebra.)
First complete the problem *Diagonalization*. In that notation:

(a) Find the matrix S whose columns are $|\alpha\rangle$ and $|\beta\rangle$. Show that $S^\dagger = S^{-1}$ by calculating $S^\dagger$ and multiplying it by S. (Does the order of multiplication matter?)

(b) Calculate $B = S^{-1}CS$. How is the matrix E related to B and C? The transformation that you have just done is an example of a "change of basis", sometimes called a "similarity transformation." When the result of a change of basis is a diagonal matrix, the process is called diagonalization.