1 Flute and boundary conditions

Adapted from Q2M.1 from Chpt 2 of Unit Q, 3rd Edition

Waves of pressure (sound waves) can travel through air. When there are boundary conditions on a sound wave, the allowed frequencies become discretized (i.e. there is a discrete set of possible values). The same thing happens in quantum mechanics with "matter waves". Before getting fully into quantum mechanics, I want to warm up with musical examples. The PDE for pressure waves in a column of air is

$$\frac{\partial^2 p}{\partial t^2} = v_{\mathrm{s}}^2 \frac{\partial^2 p}{\partial x^2} \tag{1}$$

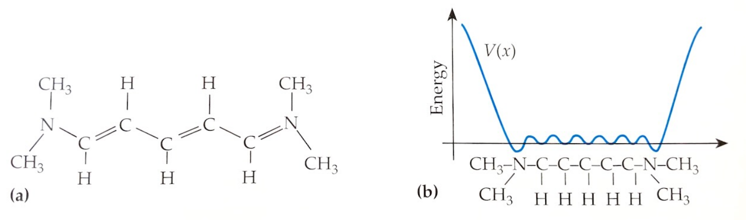
where p is the pressure at time t and position x, and v_{s} is a constant called the the speed of sound in air. We will look for solutions of the form $p(x,t) = \sin(kx)\cos(wt) + \text{constant}$. The pressure at the open end of a pipe is fixed at 1 atmosphere (this boundary condition is called a node, because pressure doesn't fluctuate). If a pipe has a closed end (which may or may not be true for a flute) the pressure at the closed end can fluctuate up and down (this boundary condition would be called an anti-node).

(a) A concert flute, shown above, is about 2 ft long. Its lowest pitch is middle C (about 262 Hz). On the basis of this evidence, should we consider a flute to be a pipe that is open at both ends, or at just one end? Support your argument with a quantitative comparison. (The end of the flute farthest from the mouth piece is clearly open. The other end of the flute seems to be closed, so if you claim that the flute is open at both ends, you should try to explain where the other open end is.)

(b) What are the lowest three frequencies that can be played on a flute when all the finger holes are closed? Give you answer in units of Hz. Draw these frequencies on a spectrogram (vertical axis represents frequency, horizontal axis is time). Multiple horizonal lines at the same time represent the superposition of multiple frequencies.

(c) **Not graded this year - think about this question if you are interested:** The orchestra is warming up their instruments. The air in the flute starts at 290 K and increases temperature to 300 K. How seriously does this affect the pitch of the flute? For reference, each step on a chromatic musical scale has a frequency 1.06 times higher than the one below it ($1.06 = 2^{1/12}$). The conductor of the orchestra will be upset if the flute shifts from its correct frequency by $\pm 1\%$. The speed of sound in a gas is $v_s = \sqrt{(\gamma P_0/\rho_0)}$ where γ is a dimensionless constant, P_0 is the ambient pressure and ρ_0 is the gas's density. As the gas warms up, the density of air inside the flute drops (the equilibrium air pressure inside the flute does not change).

2 Light from an electron in a box



Suppose an electron is trapped in a box whose length is $L = 1.2$ nm. This is a coarse-grained model for an electron in a small molecule like cyanine (see Example Q11.1 in the textbook, and the figure above). If we solve the Schrodinger equation for this coarse-grained model, the possible energy levels for this electron are

$$E = \frac{h^2 n^2}{8mL^2} \tag{2}$$

where m is the mass of the electron and $n = 1,\ 2,\ 3,\ \ldots$

Draw a spectrum chart (like the righthand side of Figure Q11.2) to show what you would see if a number of identical excited systems of this type emitted light that was dispersed by a diffraction grating.

Note: Due to the shape/symmetries of electron wavefunctions in a box, optical transitions between energy levels only happen when $\Delta n = n_{\text{initial}} - n_{\text{final}}$, is an odd integer.

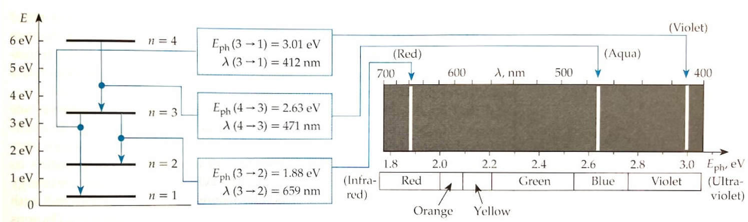


Figure Q11.2
The transitions of an electron in a 1.0-nm box that give rise to visible photons. The spectrum shown on the right is what you would see if a number of identical excited systems of this type emitted light that was dispersed by a prism.

3 Hot hydrogen atom

Find the wavelength of the photon emitted during a $n = 5 \rightarrow 4$ transition in a hydrogen atom.

Note: The energy levels in a hydrogen atom are

$$E_n = \frac{-13.6 \text{ eV}}{n^2} \tag{3}$$

where $n = 1,\ 2,\ 3,\ ...$

4 Wavelength from a charge on a spring

Suppose a charged particle is held in position by an electrostatic spring (i.e. the restoring force on the charge follows Hooke's law, $F = -kx$). The mass of the charge, and the spring constant, are such that the system has a natural frequency $\omega = 10^{16}$ rad/s (ω is a fixed parameter in this question, not a variable). Find the wavelength of the photon emitted during a $n = 1 \rightarrow 0$ transition.

By solving the Schrodinger equation for this situation, we know that the energy of the charged particle (i.e. the sum of the particle's kinetic energy, plus any potential energy stored in the spring) is given by

$$E_n = \hbar\omega(n + \frac{1}{2}) \tag{4}$$

where $n = 0,\ 1,\ 2,\ ...$

Note: Due to the shape/symmetry of the wavefunctions for particles trapped by a springlike force, optical transitions only occur when $\Delta n = \pm 1$.

Sense making: Try approaching this question from a classical physics perspective. What wavelength of light would we expect from a charge that oscillates at $\omega = 10^{16}$ rad/s?